

# A Built-in Horizontal Symmetry of $SO(10)$

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## Abstract

In a renormalizable  $SO(10)$  theory, all fermion mass matrices are linear combinations of three fundamental types,  $M^{10}$ ,  $M^{\overline{126}}$ , and  $M^{120}$ , whose superscripts indicate their  $SO(10)$  transformation properties. We point out that each of these fundamental mass matrices possesses a natural symmetry that can be used to generate an unbroken horizontal symmetry  $\mathcal{G}$ , if the natural symmetry is taken to be the residual symmetry. This built-in symmetry is a Coxeter group. If it is finite, it must be one of five groups,  $S_4$ ,  $Z_2 \times S_4$ ,  $Z_2 \times A_5$ , plus two ‘rank-4’ groups. These symmetries place constraints on the fundamental mass matrices and reduce the number of parameters in an  $SO(10)$  fit. Since they are built-in and can be derived theoretically, it is hoped that they impose better constraints than those without a theoretical basis, but that is to be confirmed because there is no attempt to fit the experimental data in this article, except to count the number of free parameters. To illustrate the similarities and differences of various kinds of constraints, a comparison is made with an existing  $S_4$  model, and with models possessing the Fritzsch texture.

## I. INTRODUCTION

The 12 fundamental fermions in nature are divided into three generations. Those in the same differ from one another by their Standard-Model quantum numbers, but there are no quantum numbers to tell the generations apart. This asymmetry, and the difficulty in identifying a horizontal symmetry, is partially due to its breaking, needed to generate mixing and to keep the masses of different generations different. Under such circumstances, not only the horizontal symmetry group has to be identified, it is also necessary to know how to break it. There are many strategies used to deal with such a task [1], one of them is to identify the natural symmetry found in the leptonic mass matrices with the residual horizontal symmetry left over after breaking. With that assumption, the unbroken horizontal symmetry can be generated from the natural symmetry [2]. In the neutrino sector, this natural symmetry is  $Z_2$ , or  $Z_2 \times Z_2$ . In the charged-lepton section, it is  $Z_k$  for an arbitrary  $k > 2$ .

Unfortunately, the horizontal symmetry for leptons [3] so obtained is very different from the symmetry for quarks [4] obtained in a similar manner. It is hard to reconcile the small mixing angles of quarks with the generally large mixing angles of neutrinos.

To ensure a common origin of symmetry for both leptons and quarks, a Grand Unified Theory (GUT) is called for. In what follows we shall take that to be  $SO(10)$  [5–8], whose irreducible representation **16** accommodates all left-handed single-colored fermions in one generation, including the heavy Majorana neutrino implicated in type-I seesaw and leptogenesis. Since  $\mathbf{16} \times \mathbf{16} = \mathbf{10} + \overline{\mathbf{126}} + \mathbf{120}$ , every fermion mass matrix in a renormalizable theory is a linear combination of three types of fundamental mass matrices,  $M^{10}$ ,  $M^{\overline{126}}$ , and  $M^{120}$ , whose superscripts indicate their  $SO(10)$  transformation property. It turns out that  $M^{10}$  and  $M^{\overline{126}}$  are symmetric matrices and  $M^{120}$  is antisymmetric. If  $a, b = 1, 2, 3$  are the generation indices, then  $M_{ab} = \pm M_{ba}$  is a relation between generation  $a$  and generation  $b$ , thus akin to a horizontal symmetry. We shall show in the next section that indeed every fundamental mass matrix has a *natural symmetry*  $(Z_2)^n$ , the direct product of  $n$   $Z_2$ 's, with some  $n$  between 1 and 7. if we identify them as *residual symmetries*, then they can be used to generate an unbroken horizontal group  $\mathcal{G}$ .

It will be shown in the next section that this built-in horizontal symmetry  $\mathcal{G}$  is a Coxeter group. Moreover, if it is finite, then it must be one of five groups. The origin of this strong result is our insistence that the residual symmetry left over after breaking is the natural

symmetry  $(Z_2)^n$ , a requirement that is not always obeyed in existing models [6, 7].

In the usual approach of a renormalizable  $SO(10)$  theory, there are three Higgs fields  $\phi^a$  ( $a = 10, \overline{126}, 120$ ) in the Yukawa terms. Vacuum expectations  $\langle \phi^a \rangle$  are assigned from which the fundamental mass matrices  $M^a$  are computed. If a horizontal symmetry  $\mathcal{G}$  is present, then the residual symmetry  $\mathcal{R}^a$  of  $M^a$  is generated by elements  $g \in \mathcal{G}$  such that  $g\langle \phi^a \rangle = \langle \phi^a \rangle$ . In the present bottom-up approach,  $\mathcal{R}^a = (Z_2)^{n_a}$  is given by the natural horizontal symmetries of  $SO(10)$ ,  $\mathcal{G}$  is generated by these  $\mathcal{R}^a$ 's, thus the equivalent vacuum alignments are invariant eigenvectors of some *order-2* elements of  $\mathcal{G}$ .

This natural symmetry in  $SO(10)$  is reminiscent of the natural symmetry  $Z_2$ , or  $Z_2 \times Z_2$ , of the neutrino mass matrix [2]. These two cases have indeed the same origin, arising from the symmetric nature of the neutrino mass matrix on the one hand, and the symmetric or antisymmetric nature of the  $SO(10)$  fundamental mass matrices on the other. In the leptonic case, the horizontal group is generated by the  $(Z_2)^n$  residual symmetry in the neutrino sector, with  $n = 1$  or  $2$ , and a  $Z_k$  residual symmetry in the charged-lepton sector, with  $k > 2$  quite arbitrary. Together they can generate an infinite number of fairly complicated finite groups that have three-dimensional irreducible representations (3dIR) appropriate to the three generations. One must comb through all of them [1–4] to fish out those whose neutrino-mixing predictions agree with data. In the  $SO(10)$  case, the residual symmetry of every fundamental mass matrix is of the type  $(Z_2)^n$ , without any  $Z_k$  for  $k > 2$ . As a result, all the finite horizontal symmetry groups  $\mathcal{G}$  they can generate are known, and among them only five possess 3dIR. This makes the search of a finite horizontal symmetry for quarks and leptons together in  $SO(10)$  much simpler than for leptons alone, though this simplicity is marred by the complexity of having to verify the validity of the vertical symmetry  $SO(10)$  at the same time.

Note that this derivation of a built-in horizontal symmetry for  $SO(10)$  relies on the symmetric or antisymmetric nature of the fundamental mass matrices, which comes about partly because all the fermions are contained in a *single* representation **16**, so it would not have worked if the GUT was  $SU(5)$ .

The natural symmetry of  $SO(10)$  together with the property of Coxeter groups will be discussed in Sec. 2. Their 3dIR will be presented in Sec. 3. The constraints a symmetry puts on the fundamental mass matrices  $M$  will be given in Sec. 4. Its application to  $SO(10)$  to determine the fermion masses and mixings will be discussed in Sec. 5, including horizontal

symmetry constraints on the fermion mass matrices and resulting number of real parameters. In Sec. 6, a comparison with an existing  $S_4$  model [7] is made, and also a comparison with models [8] possessing the Fritzsch texture [9], to illustrate the similarities and differences of various constraints. A summary is presented in Sec. 7 to conclude the article.

## II. NATURAL SYMMETRY AND COXETER GROUPS

Suppose  $M$  is a symmetric matrix, with non-degenerate eigenvalues  $m_i$ , and normalized eigenvectors  $u_i$ . By studying the matrix element  $u_j^T M u_i = m_j u_j^T u_i = m_i u_j^T u_i$ , it is easy to see that  $u_j^T u_i = 0$  if  $i \neq j$ , hence  $u_j^T u_i = \delta_{ij}$ , and  $M$  can be written in the dyadic form  $M = \sum_i m_i u_i u_i^T$ . Define  $s = \sum_i \sigma_i u_i u_i^T$  with some unknown  $\sigma_i$ . Then  $s = s^T$ , and  $s^T M s = \sum \sigma_i^2 m_i u_i u_i^T = M$  if and only if  $\sigma_i^2 = 1$  for all  $i$ . Such an  $s$  obeys  $s^2 = 1$ , and is a symmetry of  $M$ . Since each of the three  $\sigma_i$ 's can be either  $+1$  or  $-1$ , there are 8 possibilities, with one being the identity matrix. These  $s$ 's thus generate a residual symmetry group  $(Z_2)^n$ , with  $n$  between 1 and 7.

If  $M$  is antisymmetric, then  $u_j^T M u_i = m_i u_j^T u_i = -m_j u_j^T u_i$  tells us that the non-zero eigenvalues come in opposite pairs,  $(m_i, -m_i)$ . It is therefore convenient to divide the index  $i$  into two groups, with  $-a$  and  $a$  labeling the non-zero eigenvalues, so that  $-m_a = m_{-a}$ , and  $A$  labeling the zero eigenvalues. In that case the orthonormal relations of the eigenvectors become  $u_a^T u_b = \delta_{a,-b}$ ,  $u_A^T u_B = \delta_{AB}$ , and  $u_{\pm a}^T u_A = 0 = u_A^T u_{\pm a}$ . The dyadic form of  $M$  is then  $M = \sum_{i=\pm a} m_i u_i u_{-i}^T$ . Let  $s = \sum_{i=\pm a} \sigma_i u_i u_{-i}^T + \sum_A \sigma_A u_A u_A^T$ , with  $\sigma_a = +\sigma_{-a}$ . Then  $s = s^T$ , and  $s^T M s = \sum_{i=\pm a} \sigma_i^2 m_i u_i u_{-i}^T = M$  if and only if  $\sigma_{\pm a}^2 = 1$ . There is no restriction on  $\sigma_A$  but we will choose them to be either  $+1$  or  $-1$ , so that once again  $s^2 = 1$ . For  $3 \times 3$  matrices, there is only one  $a$  and one  $A$ , with  $\sigma_a = \sigma_{-a} = +1$  or  $-1$ , and  $\sigma_A = +1$  or  $-1$ . Hence the residual symmetry group of antisymmetric matrices is  $(Z_2)^n$ , with  $n = 1, 2$ .

Thus for each fundamental mass matrix  $M$  which is either symmetric or antisymmetric, one or more operators  $s = s^T$  with  $s^2 = 1$  can be found so that  $s^T M s = M$ . If we identify the natural symmetry with the residual symmetry after breaking, then the minimal unbroken horizontal symmetry group is the group generated by all these distinct  $s$ 's. Let us use a subscript to distinguish these generators, and proceed to find the structure of the group. Suppose  $s_b s_c$  has an order  $o_{bc}$ , so that  $(s_b s_c)^{o_{bc}} = 1$ . Then since  $s_b^2 = 1$ , it follows that  $s_b (s_b s_c)^{o_{bc}} s_b = s_b^2 = 1 = (s_c s_b)^{o_{bc}}$ , showing that  $o_{cb} = o_{bc}$ . Moreover,  $s_b^2 = 1$  implies

$o_{bb} = 1$ . A group generated by these ‘*simple reflections*’  $s_b$ , obeying the conditions  $o_{bb} = 1$  and  $o_{bc} = o_{cb} \geq 2$  for  $b \neq c$ , is called a *Coxeter group* [11]. The number of  $s_b$ ’s is the *rank* of the group.

A Coxeter group of rank  $n$  can be conveniently represented by a *Coxeter graph* with  $n$  nodes, each of which corresponds to a generator  $s_b$  of the group. A line is drawn connecting the pair of nodes  $b$  and  $c$  provided  $o_{bc} \geq 3$ , with the number  $o_{bc}$  written above the line if  $o_{bc} > 3$ .

If there is no line directly connecting node  $b$  and node  $c$ , then  $(s_b s_c)^2 = 1$ , which implies  $s_b s_c = s_c s_b$  because  $s_b^2 = s_c^2 = 1$ . Thus two simple reflections not directly connected mutually commute. If a Coxeter graph is disconnected, then every node in one part commute with every node in a disconnected part, so the Coxeter group is a direct product of as many Coxeter subgroups as there are disconnected parts.

All finite connected Coxeter groups are known, with most of them being Weyl groups of semisimple Lie algebras. The set of roots of a simple Lie algebra  $L$  of rank  $n$  is invariant under reflections about the hyperplane perpendicular to every simple root. The group generated by these  $n$  reflections is known as the Weyl group of the algebra, and is denoted by  $W(L)$ . Every Weyl group is a Coxeter group, with the simple reflections being the generators  $s_b$  of the Coxeter group. If  $L$  is expressed as a Dynkin diagram, then the Coxeter graph of  $W(L)$  is given by the same Dynkin diagram, with single bonds in the Dynkin diagrams corresponding to  $o_{bc} = 3$  in the Coxeter graph, double bonds to  $o_{bc} = 4$ , and triple bonds to  $o_{bc} = 5$ . The arrows do not matter so  $W(B_n) = W(C_n)$ . Weyl groups for semisimple Lie algebras are direct product of Weyl groups of simple Lie algebras.

In the literature,  $W(L)$  is often written simply as  $L$ , a convention we will adopt here. Thus, unless otherwise stated,  $A_n$  in this paper is not the Lie group  $SU(n+1)$ , nor the finite simple group  $A_n$ , nor the alternating group consisting of even permutation of  $n$  objects. It is the Weyl group  $W(A_n)$ . In this notation, the possible Weyl groups are  $A_n, B_n = C_n, D_n, G_2, F_4, E_6, E_7, E_8$ , with the subscript indicating the rank of the Coxeter group. In particular, it should be noted that  $A_1$  is simply the cyclic group  $Z_2 = S_2$ , and  $A_n$  is the symmetric group  $S_{n+1}$ .

Other than the Weyl groups, the Dihedral groups  $Dih(n)$  are rank-2 finite Coxeter groups, denoted by  $I_2(n)$ . The only other finite Coxeter groups are  $H_3$  and  $H_4$ , of ranks 3 and 4 respectively. Their Coxeter graphs are both tree graphs, with  $(o_{12}, o_{23}) = (6, 3)$  for  $H_3$ , and

$(o_{12}, o_{23}, o_{34}) = (6, 3, 3)$  for  $H_4$ .

Let us now return to  $SO(10)$  and its possible horizontal groups, generated by  $n$   $s_b$ 's. Since there are three generations of fermions, we only need to consider those groups with three-dimensional irreducible representations (3dIR). These are  $A_3, B_3, H_3, B_4, D_4$ , and no more. In terms of the Small Group (SG) designations in the GAP library [10, 12], these groups are  $A_3 = SG([24, 12]) = S_4$ ,  $B_3 = SG([48, 48]) = Z_2 \times S_4$ ,  $H_3 = SG([120, 35]) = Z_2 \times 'A'_5$ ,  $B_4 = SG([384, 5602])$ , and  $D_4 = SG([192, 1493])$ . In these expressions,  $S_4$  is the group of permutation of 4 objects, and  $'A'_5$  is the group of even permutation of 5 objects.

The Coxeter graphs for these five groups are given in Fig. 1, with the number of lines between  $b$  and  $c$  equal to  $o_{bc} - 2$ .

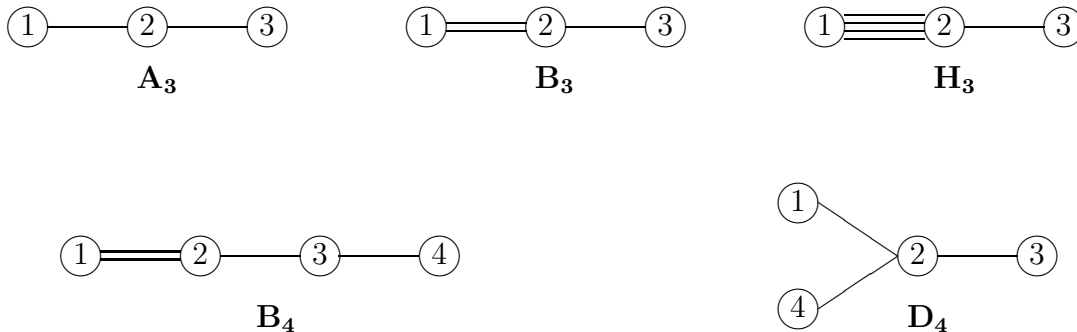


Fig. 1 Ranks 3 and 4 finite Coxeter groups with a 3dIR

### III. THREE-DIMENSIONAL IRREDUCIBLE REPRESENTATIONS

There are respectively 2, 4, 4, 4, 6 inequivalent 3dIR for  $A_3, B_3, H_3, B_4, D_4$  [10, 12], but only half of them need to be considered for the following reason. If  $\{s_b\}$  is a 3dIR of the fundamental reflections of a Coxeter group, then so is  $\{-s_b\}$ . In three dimensions, these two sets have opposite determinants, so they cannot be equivalent. However, the constraint imposed on  $M$  by  $s$  through the relation  $s^T M s = M$  is the same as the constraint imposed by  $-s$ , hence half the representations do not give rise to anything new. In what follows we will choose the representation where  $\det(s_1) = +1$ .

In  $A_3, H_3, D_4$ ,  $s_1$  and  $s_2$ , as well as  $s_2$  and  $s_3$ , are connected by a single bond, hence  $(s_1 s_2)^3 = 1$  and  $(s_2 s_3)^3 = 1$ . Thus  $\det(s_1) = +1$  implies  $\det(s_2) = +1$  and  $\det(s_3) = +1$ . This is not necessarily so for  $B_2$  and  $B_4$ , whose  $\det(s_3)$  could have either sign.

Another feature of the simply connected diagrams  $A_3, H_3, D_4$  is that none of the simple reflections  $s_i$  may be the identity matrix 1. For example, if  $s_1 = 1$ , then  $(s_1 s_2)^3 = s_2^3 = 1$ . Together with  $s_2^2 = 1$ , it yields  $s_2 = 1$ . Similarly  $s_3 = 1$ , so this representation is reducible. For that matter,  $s_1 = 1$  or  $s_3 = 1$  is not allowed in  $B_3$  either because the rank-2 graph with this node stripped off has no 3DIR, so the three-dimensional representation of  $B_3$  with  $s_1 = 1$  or  $s_3 = 1$  is not irreducible either. In fact, the only node where 1 is allowed is  $s_1$  in  $B_4$ , and the only 3DIR are those with  $s_1 = \pm 1$  and  $s_2, s_3, s_4$  form a 3DIR of  $A_3$ .

Since  $s_1$  and  $s_3$  are not directly connected in the Coxeter graphs, they commute so they can be diagonalized simultaneously. For the rank-3 groups, neither of them can be 1, nor is  $s_1 = s_3$  allowed, for otherwise the representation is essentially the same as a rank-2 group with  $s_3$  stripped, whose three-dimensional representation is reducible. For  $A_3$  and  $H_3$ , it is thus possible to choose a basis so that  $s_1 = \text{diag}(-1, -1, +1) := x$ , and  $s_3 = \text{diag}(+1, -1, -1) := z$ . For  $B_3$ , we can choose  $s_1 = x$  but  $s_3$  may be  $z$  or  $-z$ . The remaining simple reflection  $s_2$  is determined by the conditions  $(s_i s_2)^{m_{i2}} = 1$  and the result is shown in Table 1 and Eq. (1). The number  $\varphi := (1 + \sqrt{5})/2$  is the golden ratio, with  $\varphi^{-1} = \varphi - 1 = (\sqrt{5} - 1)/2$ .

| group | IR | $s_1$ | $s_2$ | $s_3$  | $s_4$ |
|-------|----|-------|-------|--------|-------|
| $A_3$ | 1  | $x$   | $y_1$ | $z$    | —     |
|       | 2  | $x$   | $y_2$ | $z$    | —     |
| $B_3$ | 1  | $x$   | $y_1$ | $z$    | —     |
|       | 2  | $x$   | $y_2$ | $-z$   | —     |
| $H_3$ | 1  | $x$   | $y_4$ | $z$    | —     |
|       | 2  | $x$   | $y_5$ | $z$    | —     |
| $B_4$ | 1  | 1     | $x$   | $y_1$  | $z$   |
|       | 2  | 1     | $-x$  | $-y_1$ | $-z$  |
| $D_4$ | 1  | $x$   | $y_1$ | $x$    | $z$   |
|       | 2  | $x$   | $y_1$ | $z$    | $x$   |
|       | 3  | $x$   | $y_1$ | $z$    | $z$   |

Table 1. Irreducible representations (IR) of the five finite Coxeter groups

Their detailed matrix forms are:

$$\begin{aligned}
y_1 &= \frac{1}{2} \begin{pmatrix} -1 & \sqrt{2} & -1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ -1 & -\sqrt{2} & -1 \end{pmatrix}, & y_2 &= \frac{1}{2} \begin{pmatrix} -1 & 1 & \sqrt{2} \\ 1 & -1 & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 0 \end{pmatrix}, & y_3 &= \frac{1}{2} \begin{pmatrix} 1 & -1 & \sqrt{2} \\ -1 & 1 & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 0 \end{pmatrix} \\
y_4 &= \frac{1}{2} \begin{pmatrix} -1 & -\varphi^{-1} & \varphi \\ -\varphi^{-1} & -\varphi & -1 \\ \varphi & -1 & \varphi^{-1} \end{pmatrix}, & y_5 &= \frac{1}{2} \begin{pmatrix} -1 & \varphi & -\varphi^{-1} \\ \varphi & \varphi^{-1} & -1 \\ -\varphi^{-1} & -1 & -\varphi \end{pmatrix}
\end{aligned} \tag{1}$$

For the rank-4 groups, as remarked earlier,  $B_4$  is obtained from the  $A_3$  representation with a  $s_1 = \pm 1$  attached. For  $D_4$ , it collapses into an  $A_3$  with either  $s_1, s_3$ , or  $s_4$  removed. With  $s_2$  given by that in  $A_3$ ,  $s_1$  fixed to be  $a$ , then  $(s_3, s_4)$  must be either  $(x, z), (z, x)$ , or  $(z, z)$ . These remarks about  $B_4$  and  $D_4$  have been incorporated in Table 1.

#### IV. CONSTRAINT ON FUNDAMENTAL MASS MATRICES

The general forms of a symmetric and an antisymmetric mass matrix are

$$M_s := \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}, \quad M_a := \begin{pmatrix} 0 & \beta & \gamma \\ -\beta & 0 & \epsilon \\ -\gamma & -\epsilon & 0 \end{pmatrix} \tag{2}$$

Table 2 gives the relations imposed on their parameters by the symmetry relation  $s^T M s = M$  for each of the  $s$  in Table 1. If  $s = 1$ , then there is no constraint whatsoever, enabling  $M_s$  to be any complex symmetric matrix and  $M_a$  to be any complex antisymmetric matrix.

| $s$   | $M_s$  | $M_a$  |
|-------|--|--|
| 1     | —  | —  |
| $x$   | $c = e = 0$  | $\gamma = \epsilon = 0$  |
| $z$   | $b = c = 0$  | $\beta = \gamma = 0$   |
| $y_1$ | $c = -d + (a + f)/2, e = -b - (a - f)/\sqrt{2}$  | $\gamma = \sqrt{2}\beta, \epsilon = -\beta$                      |
| $y_2$ | $b = f - (a + d)/2, c = e - (a - d)/\sqrt{2}$  | $\gamma = -\beta/\sqrt{2}, \epsilon = \beta/\sqrt{2}$            |
| $y_3$ | $b = f - (a + d)/\sqrt{2}, c = e + (a - d)/\sqrt{2}$   | $\gamma = \beta/\sqrt{2}, \epsilon = -\beta/\sqrt{2}$            |
| $y_4$ | $c = b + [-(\varphi + \varphi^{-1})a + (\varphi^{-1} - 1)d + (\varphi + 1)f]/2$<br>$e = -b + [\varphi^{-1}a + d - \varphi f]/2$  | $\gamma = (1 - \varphi^{-1})\beta, \epsilon = \varphi^{-1}\beta$ |
| $y_5$ | $c = b + [-(\varphi + \varphi^{-1})a - (\varphi + 1)d - (\varphi^{-1} - 1)f]/2$<br>$e = -b + [-\varphi a + d + \varphi^{-1}f]/2$ | $\gamma = (1 + \varphi)\beta, \epsilon = -\varphi\beta$          |



Table 2. Symmetry constraints on symmetric  $M_s$  and antisymmetric  $M_a$  mass matrices

## V. FERMION MASS MATRICES

Since every fermion is contained in **16**, the fermion mass matrices  $m_\alpha$  ( $\alpha = u, d, e, \nu$ ) can be obtained from the  $16 \times 16$  fundamental mass matrices  $M^{10} \equiv H$ ,  $M^{120} \equiv G$  and  $M^{\overline{126}} \equiv F$ .  $H$  contributes equally to  $m_\nu$  and  $m_u$ , and equally to  $m_e$  and  $m_d$ , whereas  $F$  contributes  $-3$  times as much to  $m_\nu$  as  $m_u$ , and  $-3$  times as much to  $m_e$  as  $m_d$ . Only  $\overline{126}$  contains a Standard-Model singlet, so the Majorana mass matrices receives a contribution only from  $F$ . The effective mass matrix for the active neutrinos is obtained from the neutrino Dirac mass matrix  $m_\nu$  and the Majorana mass matrices  $m_R$  and  $m_L$  by the formula

$$\overline{m}_\nu = -m_\nu m_R^{-1} m_\nu^T + m_L, \quad (3)$$

where the first term comes from the type-I seesaw mechanism and the second term comes from the type-II seesaw.

These relations between fermion mass matrices and fundamental mass matrices are summarized in Table 3, where  $r_i$  are arbitrary coefficients. The normalization of  $H, F$  and  $G$  is determined by choosing the coefficients of  $m_d$  in all of them to be 1.

|         | $H$   | $F$     | $G$   |
|---------|-------|---------|-------|
| $m_u$   | $r_1$ | $r_2$   | $r_3$ |
| $m_d$   | 1     | 1       | 1     |
| $m_e$   | 1     | $-3$    | $r_4$ |
| $m_\nu$ | $r_1$ | $-3r_2$ | $r_5$ |
| $m_R$   | 0     | $r_6$   | 0     |
| $m_L$   | 0     | $r_7$   | 0     |

Table 3. Relations between fermion and fundamental mass matrices

There are currently  $18=13+5$  experimentally measured values associated with the fermion mass matrices, in which 5 are neutrino quantities and 13 are non-neutrino. The neutrino ones are the two oscillation mass gaps, and the three PMNS mixing angles. The others are the nine charged-fermion masses and the four CKM mixing parameters.

In general, both the fundamental mass matrices  $H, F, G$  and the coefficients  $r_i$  are complex, though phases may be chosen to render one  $r_i$  per fermion mass matrix real. Together

they contain many more parameters than the available experimental quantities, so various ways have been devised in the literature [5–8] to reduce the number of parameters to be close to the experimental number of 18. Dropping all contributions from  $G$  is one way. Another way is to assume the fundamental and the fermion mass matrices to be hermitian, hence all the coefficients  $r_i$  to be real. This assumption can be justified if CP symmetry is broken spontaneously, a theory sometimes referred to as the charge-conjugation-conservation (CCC) [13] theory. Since  $H$  and  $F$  are hermitian and symmetric, their matrix elements are real, thus each is described by 6 (real) parameters.  $G$  is hermitian and antisymmetric, hence its matrix elements are purely imaginary, with 3 parameters. From Table 3, we see that there are 7  $r_i$ 's (6 if only one of type-I and type-II seesaw is present). Altogether there are 22 parameters, still larger than the 18 available experimentally, thus more constraints can be imposed.

Horizontal symmetry is another way to reduce the number of parameters [6]. With a built-in finite symmetry, it must be either  $A_3, B_3, H_3, B_4$ , or  $D_4$ . Each fundamental mass matrix  $M$  must be invariant under a simple reflection generator  $s$  of the group, but its  $SO(10)$  transformation property is up to us to choose. For example, for rank-3 groups, we can assign the three of them to transform like  $H, F, G$  respectively, or we can assign two of them to transform like  $H$ , and one like  $F$ , etc. Since there are two constraints per simple reflection, in the first case we reduce the total parameters of  $H, F, G$  from 15 to 9, yielding a total of 16 parameters in a CCC theory, two short of the experimental quantities. If that fits well, it is a strong indication of the validity of the horizontal symmetry. For rank-4 groups, at least two of  $M$ 's must have the same  $SO(10)$  transformation property, which tends to increase the number of available parameters compared to the rank-3 groups, but what that is depends on the details.

All in all, there are many ways to assign the horizontal and vertical transformation properties of the fundamental mass matrices, thereby producing many possible models even for a single horizontal group. A systematic attempt to cover all possibilities involves a large amount of work, but the amount is finite because there are only five possible groups. For each fit, we must use experimental values extrapolated to GUT energy, and that depends on the detailed dynamics in between, which further adds to the complication. Since the five horizontal symmetries are built into  $SO(10)$  and theoretically derived, it is hoped that the constraints they provide would be better than those without a strong theoretical basis.

However, we will not attempt any of these fits in the present article.

It should be mentioned that in the discussion above, we implicitly assumed that every  $M$  has a single  $Z_2$  symmetry. Recall however that the symmetry could be  $Z_2 \times Z_2$ . In that case there would be three or four constraints for the matrix elements of  $M$ , rather than just two.

In the opposite direction, we may assign two  $M$ 's with the same  $SO(10)$  transformation to be invariant under different simple reflections,  $s_i$  and  $s_j$ . The end result is like having only one  $M$ , but with fewer constraints on its matrix elements. For example, if  $M_x$  is invariant under  $x$  of Table 1 and  $M_z$  is invariant under  $z$ , and both are of type  $H$ , then their sum is still of type  $H$ , and according to Table 2 it is of the form

$$M := M_x + M_z = \begin{pmatrix} a & b & 0 \\ b & d & 0 \\ 0 & 0 & f \end{pmatrix} + \begin{pmatrix} a' & 0 & 0 \\ 0 & d' & e' \\ 0 & e' & f' \end{pmatrix}. \quad (4)$$

The result is a symmetric matrix with the (13) and (31) elements zero, and no further constraint on any of the other matrix elements. As another example, consider  $M = M_{y_1} + M_z$ . Then

$$M := M_{y_1} + M_z = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} + \begin{pmatrix} a' & 0 & 0 \\ 0 & d' & e' \\ 0 & e' & f' \end{pmatrix}, \quad (5)$$

where  $c = -d + (a + f)/2$  and  $e = -b - (a - f)/\sqrt{2}$ . The result is a symmetric matrix with no constraint whatsoever on any of its elements. The same would be true for the sum  $M = M_{y_i} + M_z$  for  $i = 2, 3, 4, 5$ .

## VI. HORIZONTAL SYMMETRY AND OTHER CONSTRAINTS

To compare the use of built-in horizontal symmetry to impose constraints with other approaches in the literature, we discuss two specific examples in this section as an illustration.

### A. $S_4$

An interesting  $SO(10)$  model possessing an  $S_4$  horizontal symmetry is given in Ref. [7]. The fundamental mass matrices in that model are [14]

$$H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \tilde{M} \end{pmatrix}, \quad H' = \begin{pmatrix} 0 & \delta & -\delta \\ \delta & 0 & 0 \\ -\delta & 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & m_1 & m_1 \\ m_1 & m_0 & m_1 - m_0 \\ m_1 & m_1 - m_0 & m_0 \end{pmatrix}, \quad (6)$$

and  $G = 0$ , where  $H'$  has the same  $SO(10)$  transformation property as  $H$ . The parameter  $\tilde{M}$  is real, and  $\delta, m_0, m_1$  are complex.

Since  $A_3 = S_4$  is one of the five built-in symmetries, the success of this model seems to confirm their presence. Unfortunately this is not so because the residual symmetry left behind after the breaking of the  $S_4$  in Ref. [7] is not the simple reflection generators  $s_1, s_2, s_3$  of  $A_3$ . Thus whether the built-in  $A_3$  is a symmetry or not must be decided by a new fit.

To see this point in more detail, let us express the generators  $s_1 = x$ ,  $s_2 = y_1$ ,  $s_3 = z$  of Table 1 in a basis that gives rise to  $F$  in (6). This is accomplished by making a similarity transformation using

$$U = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{2} & 2 & 0 \\ \sqrt{2} & -2 & \sqrt{3} \\ \sqrt{2} & -1 & -\sqrt{3} \end{pmatrix}, \quad (7)$$

to get the generators  $x' = UxU^T$ ,  $y'_1 = Uy_1U^T$ , and  $z' = UzU^T$  in the new basis:

$$x' = - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad y'_1 = \frac{1}{4} \begin{pmatrix} 2 & -\sqrt{6} & \sqrt{6} \\ -\sqrt{6} & -3 & -1 \\ \sqrt{6} & -1 & -3 \end{pmatrix}, \quad z' = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}. \quad (8)$$

The invariant conditions  $s^T M_s s = M_s$  for the symmetric matrix  $M_s$  in (2) can be worked out to be:

1. for  $s = x'$ :  $c = b$ ,  $f = d$ ;
2. for  $s = y'_1$ :  $c = \sqrt{6}(d - a - e) - 5b$ ,  $f = 6(a + e) - 5d + 4\sqrt{6}b$ ;
3. for  $s = z'$ :  $c$  and  $f$  are determined by the conditions  $a + b + c = b + d + e = c + e + f$ .

In other words,  $M_{x'}$  is 2-3 symmetric and  $M_{z'}$  is magic. Since  $F$  in (6) is 2-3 symmetric and magic, it can be obtained either from  $M_{x'}$  or  $M_{z'}$ . The other two  $M$ 's must then be equal to

$H$  and  $H'$  in (6), if the  $S_4$  breaking in [7] respects the residual invariants of  $x', y'_1, z'$ . There is no problem in getting  $H$  but it is not possible to get  $H'$ . This shows that the residual symmetry for (6) is not  $x', y'_1, z'$ .

## B. Fritzsche Texture

We saw in the last section that the CCC theory contains 22 parameters, still more than the 18 experimental quantities available. One way to reduce the parameters further is to assume every fundamental and fermion mass matrices to have the Fritzsche texture [9]. That is, to assume not only that they are hermitian, but also that their (11), (13), and (31) matrix elements vanish. This cuts down 5 more parameters to a total of 17, one short of the experimental quantities. Reasonable fits are reported in such a scheme [8].

We saw in the last section that the CCC theory with built-in horizontal symmetry groups of rank-3 has 16 parameters. If we go to rank-4 groups, then the number of parameters increase. For example, in  $D_4$ , if we assign the fourth mass matrix to be a  $G$ -type, then two more parameters are added to make it 18: one in the matrix element of the new  $G$ , and one each from the coefficients for  $m_u$  and  $m_\nu$  for this new  $G$ . If we assign the fourth fundamental mass matrix to be of  $H$  or  $F$  type, then even more free parameters are available. If we use  $B_4$ , since  $s_1 = 1$  does not place any constraint on the fundamental mass matrices, there are more parameters still. All in all, there seems to be a sufficient number of parameters to make a successful experimental fit quite possible.

There is some formal similarity between the horizontal-symmetry constraints in Tables 1 and 2 on the one hand, and the Fritzsche-texture constraint on the other. First of all, both place two constraints on each of the fundamental mass matrices. Secondly, we see in Table 1 that  $x$  and  $z$  are common generators for all the groups. If we let  $M_x$  to be  $H$  type and  $M_z$  to be  $F$  type, then we see in Table 2 that both their (13)=(31) matrix elements vanish, just like in the Fritzsche texture. In addition, for  $M_x$ , instead of having (11) zero as in the Fritzsche texture, it is (23)=(32) that is zero. For  $M_z$ , instead of having (11) zero, it is (12)=(21) that is zero. Moreover, in the case of  $D_4$ , we can always assign another  $x$  or  $z$  to  $G$  to make its (13) element  $\gamma$  vanish as well, as in Fritzsche texture. The main difference with the Fritzsche texture is that the zeros of the latter are in fixed positions for all fundamental matrices, but that is not the case for the built-in symmetries.

## VII. SUMMARY

The main purpose of this article is to point out that there is a built-in horizontal symmetry for  $SO(10)$ , in the form of a Coxeter group. For general fundamental mass matrices without any constraint, that Coxeter group is infinite in size. If we demand the symmetry group to be finite, then it is limited to only five groups. This result is based on the reasonable assumption that natural symmetries are the residual symmetries left behind after breaking, an assumption already used fairly widely in analyzing neutrino physics.

Some immediate consequences of this conclusion are discussed. This includes how the constraints from such horizontal symmetries reduce the number of free parameters used to fit the data. The details of these constraints are quite different from those used in the literature. This point is illustrated in the last section in an  $S_4$  model, and for the Fritzsch texture.

Since finite built-in horizontal symmetries for  $SO(10)$  can be derived, it is hoped that they can offer better constraints than those without a theoretical basis. However, at present that remains only a hope because no attempt has been made to fit the data in this article. This important task of fitting will be left to future research.

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not to be confused with the symbol for the fundamental mass matrix.